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STRUCTURE FROM MOTION: AN
AUGMENTED PROBLEM AND A NEW ALGORITHM

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**Structure From Motion:
An Augmented Problem and a New Algorithm**

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ABSTRACT

Structure From Motion has been studied by S. Ullman [1]. The basic result is that given three views of four noncoplanar points, the structure (the relative depth of these points) can be uniquely determined. The original algorithm was complicate to implement. In this study, We augment the problem to include the unknown scales among these frames and a new computational algorithm which requires only linear computation is introduced. Detailed examples are provided to illustrate the method.

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1. Introduction

Analysis and processing of sequence of images has received extensive research attention in recent years. The estimation of object motion, tracking, object structure and segmentation of objects are some of its applications.

One direction of research is based on well separated feature points as observables. Using the rigid transformation, one can relate the image coordinates to underlying motion and structure of the object. Ullman [1] showed that one can uniquely recover the 3-D structure and underlying motion from three views of four non-coplanar points. The computation is complex and nonlinear.

In this paper, we deal with an augmented problem where the scale between these three views are unknown. A new computational method based on elementary matrix theory is presented to solve this problem. The computational process is fast and efficient. Detailed examples are provided to illustrate the theory. This technique is also well suited to error analyses.

2. Problem statements

Throughout the paper, orthographic projection will be assumed. The movement of the object in these views may be attributed to the motion of camera or object or both. The coordinate system will be chosen to coincide with the natural coordinate system associated with the first view. This means that x-axis will be the horizontal axis of the first image; y-axis will be the vertical axis of the first image; and the viewing (optical) direction will be taken as z-axis. The problem is: how to compute motion between frames and derive the relative depth of the object points, given three views with unknown scale among frames as depicted in Figure 1,

A general motion can be decomposed into a rotation followed by a translation. In the case of orthographic projection, it is trivial to compute translational components once the correspondence of points is established. One can choose a reference point, then the

horizontal and the vertical components of translation are simply the displacement of the reference point in the two frames. The depth component of translation is inherently lost. Thus, we essentially deal with rotations.

A rotation can be described by a rotational axis and a rotational angle. Further, two parameters - tilt and slant - are often used to describe a 3D unit vector. Tilt is the angle between the horizontal axis (on the image plane) and the projection of the 3D vector in the image plane. Thus, tilt can range from zero to 360 degrees. The slant is the angle between the 3D-vector and the optical axis (or viewing direction). Thus, the slant can range from 0 (lies on the optical axis) to 90 degrees (lies above eyes). The third parameter is the rotational angle about the axis which can be 0 to 360 degrees. In the case of zero or 360 degrees, there is no motion. It was shown in [6] that a rotational matrix can be written as:

$$R = \begin{bmatrix} n_1^2 + (1 - n_1^2)\cos\theta & n_1n_2(1 - \cos\theta) - n_3\sin\theta & n_1n_3(1 - \cos\theta) + n_2\sin\theta \\ n_1n_2(1 - \cos\theta) + n_3\sin\theta & n_2^2 + (1 - n_2^2)\cos\theta & n_2n_3(1 - \cos\theta) - n_1\sin\theta \\ n_1n_3(1 - \cos\theta) - n_2\sin\theta & n_2n_3(1 - \cos\theta) + n_1\sin\theta & n_3^2 + (1 - n_3^2)\cos\theta \end{bmatrix}$$

where $(n_1 \ n_2 \ n_3)$ is the unit vector along the axis and θ is the rotational angle.

We also write R as below for convenience of notation.

$$\begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & r_{31} \\ \cdot & \cdot & r_{32} \\ r_{13} & r_{23} & \cdot \end{bmatrix} = [r_1 \ r_2 \ r_3]$$

where $(r_{31} \ r_{32})^t$ and $(r_{13} \ r_{23})^t$ are called directional vectors. To see the meaning of directional vector, one can perform the following observation. Given a point $(a \ b \ s)$ where the depth component s is unknown. The new position of this point attributed to motion will be $R^* (a \ b)^t + s (r_{31} \ r_{32})$ where R^* is the minor of R . This is equivalent to say that the new position would lie in the line passing through $R^* (a \ b)^t$ with direction $(r_{31} \ r_{32})$. The other directional vector has a similar role if one interchanges the first frame and the second frame.

The rotation which transforms the first frame to the second frame will be denoted by S ; and that between the second and the third frame will be denoted by R ; and the rotation between the first and the third frame will be denoted by T (see Figure 2). We can write $T = R S$ or

$$\begin{bmatrix} t_{31} \\ t_{32} \\ t_{13} \ t_{23} \ . \end{bmatrix} = \begin{bmatrix} r_{31} \\ r_{32} \\ r_{13} \ r_{23} \ . \end{bmatrix} \begin{bmatrix} s_{31} \\ s_{32} \\ s_{13} \ s_{23} \ . \end{bmatrix}$$

If directional vectors are null vector, then the effect is essentially two dimensional rotation or reflection. This case (the observed image sequences) does not yield any new information about the structure of object would be excluded and the motion would be called degenerate.

3. Related Work

In [1], he showed that "object structure can be recovered if three views of four non-coplanar points are observed" for the case of orthographic projection. Also, Ullman derived a set of "polar" equations for relating object points and motion parameters. In [2], Nagel derived a set of compact nonlinear equations which specifies possible 3D rotations of rigid objects, compatible with the measurements of five object points in two views for the case of perspective projection. It was suggested to deriving the 3D rotation first and solve the translation afterwards. In [3], they derive 20 nonlinear equations relating the image space coordinates and the camera position parameters for the case of perspective projection. Numerical techniques for solving nonlinear equations are mentioned although no results are reported.

In [4], a two-stage method was introduced for the problem of two views of eight points. First one computes "eight pure parameters" from eight or more image space points. Once the eight pure parameters are obtained, the motion parameters can be obtained by solving a sixth-degree equation in one variable. Simulations are performed with the result of high sensitivity. In [5], the details of experiments on estimating the 3D motion parameters of a rigid body from two consecutive images are reported. However, satisfactory results can only be obtained by restricting a very small rotational angle (between 1 to 5 degrees) to the motion.

In this paper, we augment the problem of three views of four noncoplanar points to include the possibility of unknown scales between the frames. We present a new computational method to such a problem. The theoretical error analysis for this technique can be anticipated and is currently under investigation.

4. Method

In this section, we will first deal with three views of the same scale. There are two steps: Theorem 1 computes the tilt direction from observables of any two views and the

second step is to derive the slant and the rotational angle through the derived rotational matrix. The second step requires three observations. One observation is that the directional vectors can be derived. Another observation is that a vector perpendicular to the vector formed by $(s_1 s_2 s_3)$ where s_i is the depth component of A_i can be computed. The last observation is that one can derive the coordinates of the two unit vectors - $(1 0 0)$ and $(0 1 0)$ - in terms of basis $\{A_1, A_2, A_3\}$ which are assumed to be noncoplanar. Once we have done this, the modification of the computational algorithm to adapt to the factor of the unknown scale will be discussed. The result here is that one might have three extra solutions other than the original one.

Theorem 1: Let M be a point on the rotational axis, and $\bar{M} = (m_1 m_2)^t$ be its projection in the image plane. Then $(m_1 m_2)^t$ can be derived up to a scalar.

Proof: Since O, A_1, A_2, A_3 are not coplanar, we can take $\{A_1, A_2, A_3\}$ as a basis for 3D space. Therefore, there exist unique scalars $\alpha_1, \alpha_2, \alpha_3$ such that

$$\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 = M \quad (1)$$

Apply the rotation R , we have

$$\alpha_1 B_1 + \alpha_2 B_2 + \alpha_3 B_3 = M \quad (2)$$

Examining the first two components of (1) and (2), we get

$$\alpha_1 \bar{A}_1 + \alpha_2 \bar{A}_2 + \alpha_3 \bar{A}_3 = \bar{M} = (m_1 m_2)^t \quad (3)$$

$$\alpha_1 \bar{B}_1 + \alpha_2 \bar{B}_2 + \alpha_3 \bar{B}_3 = \bar{M} = (m_1 m_2)^t \quad (4)$$

Rewrite (3)(4) into matrix form, we have

$$\begin{bmatrix} \bar{A}_1 & \bar{A}_2 & \bar{A}_3 \\ \bar{B}_1 & \bar{B}_2 & \bar{B}_3 \end{bmatrix}_{4 \times 3} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} m_1 \\ m_2 \\ m_1 \\ m_2 \end{bmatrix} \quad (A)$$

To have a solution, the rank of the following augmented matrix must not exceed the rank

of the above 4×3 matrix. Thus, the determinant must be zero.

$$\begin{bmatrix} \bar{A}_1 & \bar{A}_2 & \bar{A}_3 & \bar{M} \\ \bar{B}_1 & \bar{B}_2 & \bar{B}_3 & \bar{M} \end{bmatrix}_{4 \times 4}$$

Compute its determinant as follows:

$$\begin{vmatrix} a_{11} & a_{21} & a_{31} & m_1 \\ a_{12} & a_{22} & a_{32} & m_2 \\ b_{11} & b_{21} & b_{31} & m_1 \\ b_{12} & b_{22} & b_{32} & m_2 \end{vmatrix} = -m_1 \begin{vmatrix} a_{12} & a_{22} & a_{32} \\ b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \end{vmatrix} + m_2 \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \end{vmatrix} \\ -m_1 \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ b_{12} & b_{22} & b_{32} \end{vmatrix} + m_2 \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ b_{11} & b_{21} & b_{31} \end{vmatrix}$$

It is obvious that the determinants of the above 3×3 can be computed which leads to

$$-m_1 a + m_2 b = 0$$

where

$$a = \begin{vmatrix} a_{12} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ b_{12} & b_{22} & b_{32} \end{vmatrix} + \begin{vmatrix} a_{12} & a_{22} & a_{32} \\ b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \end{vmatrix} = \text{tem 1} + \text{tem 2} \quad (\text{A.1})$$

$$b = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ b_{11} & b_{22} & b_{32} \end{vmatrix} = \text{tem 4} + \text{tem 3} \quad (\text{A.2})$$

Thus the tilt direction is $(b \ a)$. Note that a, b must not be simultaneous zero to have such a conclusion. This case is addressed in the appendix. **Q.E.D.**

The following outlines the summarized strategy to deal with the second step.

1. Lemma 1 computes directional vectors for $R \ S \ T$ up to a scalar.
2. The ratio of the magnitude of these directional vectors will be derived.
3. A vector perpendicular to the depth vector formed by the depth components of the observables will be derived.

4. The rotational matrix will be derived. The slant and rotational angle can then be easily derived.

Lemma 1: The following formulas holds:

$$\begin{aligned} s_{31} &= \eta \begin{vmatrix} \bar{A}_1 & \bar{A}_2 & \bar{A}_3 \\ b_{11} & b_{21} & b_{31} \end{vmatrix}; \quad s_{32} = \eta \begin{vmatrix} \bar{A}_1 & \bar{A}_2 & \bar{A}_3 \\ b_{12} & b_{22} & b_{32} \end{vmatrix}; \quad t_{31} = \begin{vmatrix} \bar{A}_1 & \bar{A}_2 & \bar{A}_3 \\ c_{11} & c_{21} & c_{31} \end{vmatrix}; \quad t_{32} = \begin{vmatrix} \bar{A}_1 & \bar{A}_2 & \bar{A}_3 \\ c_{12} & c_{22} & c_{32} \end{vmatrix}; \\ s_{13} &= \delta \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ \bar{B}_1 & \bar{B}_2 & \bar{B}_3 \end{vmatrix}; \quad s_{23} = \delta \begin{vmatrix} a_{12} & a_{22} & a_{32} \\ \bar{B}_1 & \bar{B}_2 & \bar{B}_3 \end{vmatrix}; \quad r_{31} = \begin{vmatrix} c_{11} & c_{21} & c_{31} \\ \bar{B}_1 & \bar{B}_2 & \bar{B}_3 \end{vmatrix}; \quad r_{32} = \begin{vmatrix} c_{21} & c_{22} & c_{32} \\ \bar{B}_1 & \bar{B}_2 & \bar{B}_3 \end{vmatrix}; \\ r_{13} &= \begin{vmatrix} \bar{C}_1 & \bar{C}_2 & \bar{C}_3 \\ b_{11} & b_{21} & b_{31} \end{vmatrix}; \quad r_{23} = \begin{vmatrix} \bar{C}_1 & \bar{C}_2 & \bar{C}_3 \\ b_{12} & b_{22} & b_{32} \end{vmatrix}; \quad t_{13} = \begin{vmatrix} \bar{C}_1 & \bar{C}_2 & \bar{C}_3 \\ a_{11} & a_{21} & a_{31} \end{vmatrix}; \quad t_{23} = \begin{vmatrix} \bar{C}_1 & \bar{C}_2 & \bar{C}_3 \\ a_{12} & a_{22} & a_{32} \end{vmatrix}; \end{aligned}$$

where η, δ are two unknown scalars; and the ratio of η and δ is known (notice that one knows the magnitude and the sign of this ratio). Scalars associated with other terms such as $t_{ij}'s, r_{ij}'s$ are not listed here.

Proof: Formulas for $s_{ij}'s$ are shown here. The others follow similarly. Since vectors A_1, A_2, A_3 are noncoplanar, there exists unique numbers $\alpha_1, \alpha_2, \alpha_3$ such that

$$\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 = (0 \ 0 \ 1)^t \quad (5)$$

Applying rotation S , it gives

$$\alpha_1 B_1 + \alpha_2 B_2 + \alpha_3 B_3 = S (0 \ 0 \ 1)^t = s_3 \quad (6)$$

Examining the first two components of (5)(6), one has

$$\begin{bmatrix} \bar{A}_1 & \bar{A}_2 & \bar{A}_3 \\ \bar{B}_1 & \bar{B}_2 & \bar{B}_3 \end{bmatrix}_{4 \times 3} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ s_{31} \\ s_{32} \end{bmatrix} \quad (B)$$

In order to have a solution, the following determinant must be zero.

$$\begin{vmatrix} a_{11} & a_{21} & a_{31} & 0 \\ a_{12} & a_{22} & a_{32} & 0 \\ b_{11} & b_{21} & b_{31} & s_{31} \\ b_{12} & b_{22} & b_{32} & s_{32} \end{vmatrix} = -s_{31} \begin{vmatrix} \bar{A}_1 & \bar{A}_2 & \bar{A}_3 \\ b_{12} & b_{22} & b_{32} \end{vmatrix} + s_{32} \begin{vmatrix} \bar{A}_1 & \bar{A}_2 & \bar{A}_3 \\ b_{11} & b_{21} & b_{31} \end{vmatrix} = 0$$

Thus, up to an unknown scalar η , one derives (again the coefficients must not be zero simultaneously, this could not occur and shown in the appendix).

$$s_{31} = \eta \begin{vmatrix} \bar{A}_1 & \bar{A}_2 & \bar{A}_3 \\ b_{11} & b_{21} & b_{31} \end{vmatrix} ; s_{32} = \eta \begin{vmatrix} \bar{A}_1 & \bar{A}_2 & \bar{A}_3 \\ b_{12} & b_{22} & b_{32} \end{vmatrix}$$

Using the same reasoning, there exists unique numbers $\alpha_1, \alpha_2, \alpha_3$ such that

$$\alpha_1 B_1 + \alpha_2 B_2 + \alpha_3 B_3 = (0 \ 0 \ 1)^t \quad (7)$$

Applying rotation S' (motion from second frame to the first frame), it gives

$$\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 = S' (0 \ 0 \ 1)^t = (s_{13} \ s_{23} \ s_{33})^t \quad (8)$$

Examining the first two components of (7)(8), one has

$$\begin{bmatrix} \bar{A}_1 & \bar{A}_2 & \bar{A}_3 \\ \bar{B}_1 & \bar{B}_2 & \bar{B}_3 \end{bmatrix}_{4 \times 3} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} s_{13} \\ s_{23} \\ 0 \\ 0 \end{bmatrix} \quad (E)$$

In order to have a solution, the following determinant must be zero.

$$\begin{vmatrix} a_{11} & a_{21} & a_{31} & s_{13} \\ a_{12} & a_{22} & a_{32} & s_{23} \\ b_{11} & b_{21} & b_{31} & 0 \\ b_{12} & b_{22} & b_{32} & 0 \end{vmatrix} = -s_{13} \begin{vmatrix} a_{12} & a_{22} & a_{32} \\ \bar{B}_1 & \bar{B}_2 & \bar{B}_3 \end{vmatrix} + s_{23} \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ \bar{B}_1 & \bar{B}_2 & \bar{B}_3 \end{vmatrix} = 0$$

Thus, up to a scalar δ , one derives

$$s_{13} = \delta \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ \bar{B}_1 & \bar{B}_2 & \bar{B}_3 \end{vmatrix} ; s_{23} = \delta \begin{vmatrix} a_{21} & a_{22} & a_{32} \\ \bar{B}_1 & \bar{B}_2 & \bar{B}_3 \end{vmatrix}$$

Since $s_{31}^2 + s_{32}^2 = s_{13}^2 + s_{23}^2$, one can derive the ratio of $|\eta|$ and $|\delta|$. Furthermore, the vector $(s_{31} \ s_{32}) + (s_{13} \ s_{23})$ must be parallel to the tilt direction, thus the sign can be determined. Notice that one could not decide the sign if the unknown scale is introduced since tilt direction is not available from theorem 1 in this case. Further, this lemma is still valid for the case of a different scale. Lemma 1 is also the reason that one may have three extra solutions. **Q.E.D**

When scales are different among the three frames, the tilt direction could not be obtained from theorem 1. Furthermore the four possible pairs (sign consideration) of $(s_{31} \ s_{32})$ and $(s_{13} \ s_{23})$ could not be resolved as stated in the Lemma 1. However, the vector $(s_{31} \ s_{32}) + (s_{13} \ s_{23})$ gives the tilt direction of the underlying motion. Now one has four possible pairs which would yield two possible tilt directions. Suppose that the inverse of the relative scale of the second frame with respect to the first frame is σ . The tilt direction can be obtained from (see A.1 A.2)

$$a = \sigma \text{ tem } 1 + \sigma^2 \text{ tem } 2 ; \text{ and } b = \sigma \text{ tem } 4 + \sigma^2 \text{ tem } 3 ;$$

where $\text{tem } 1, \text{tem } 2, \text{tem } 3, \text{tem } 4$ are calculated as (A1) and (A2). Actually $(\text{tem } 4 + \sigma \text{ tem } 3, \text{tem } 1 + \sigma \text{ tem } 2)$ is the tilt direction. One could easily solve for σ where σ must be positive by equating the vector $(s_{31} \ s_{32}) + (s_{13} \ s_{23})$ and $(\text{tem } 4 + \sigma \text{ tem } 3, \text{tem } 1 + \sigma \text{ tem } 2)$. Once the scale is recovered, the same technique can be employed.

The second observation is quite interesting and subtle. Since $B = SA$, it gives $Bu = SAu$ for all $u \in R^3$. Choose u such that u is perpendicular to the last row of A and B . One derives that

$$\begin{bmatrix} b_{11} u_1 + b_{21} u_2 + b_{31} u_3 \\ b_{12} u_1 + b_{22} u_2 + b_{32} u_3 \\ 0 \end{bmatrix} = \begin{bmatrix} s_{11} & s_{21} & s_{31} \\ s_{12} & s_{22} & s_{32} \\ s_{13} & s_{23} & s_{33} \end{bmatrix} \begin{bmatrix} a_{11} u_1 + a_{21} u_2 + a_{31} u_3 \\ a_{12} u_1 + a_{22} u_2 + a_{32} u_3 \\ 0 \end{bmatrix}$$

Up to a scalar, clearly one can derive that (note that s_{13}, s_{23} can not be both zero simultaneously due to nondegenerate motion)

$$\begin{bmatrix} a_{11} u_1 + a_{21} u_2 + a_{31} u_3 \\ a_{12} u_1 + a_{22} u_2 + a_{32} u_3 \end{bmatrix} = \begin{bmatrix} -s_{23} \\ s_{13} \end{bmatrix} \text{ or } = \begin{bmatrix} s_{23} \\ -s_{13} \end{bmatrix}$$

and

$$\begin{bmatrix} b_{11} u_1 + b_{21} u_2 + b_{31} u_3 \\ b_{12} u_1 + b_{22} u_2 + b_{32} u_3 \end{bmatrix} = \begin{bmatrix} -s_{32} \\ s_{31} \end{bmatrix} \text{ or } = \begin{bmatrix} s_{32} \\ -s_{31} \end{bmatrix}$$

In other words, we can write it into

$$\begin{bmatrix} \bar{A}_1 & \bar{A}_2 & \bar{A}_3 \\ \bar{B}_1 & \bar{B}_2 & \bar{B}_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -s_{23} \\ s_{13} \\ -s_{32} \\ s_{31} \end{bmatrix} \text{ or } \begin{bmatrix} -s_{23} \\ s_{13} \\ s_{32} \\ -s_{31} \end{bmatrix}$$

It is obvious that this linear system of three equations and four variables must be consistent. In other words, the determinant of this 4×3 matrix augmented by the right hand side must be zero. If one actually manipulates (see appendix), one would find out that the following is the only choice.

$$\begin{bmatrix} \bar{A}_1 & \bar{A}_2 & \bar{A}_3 \\ \bar{B}_1 & \bar{B}_2 & \bar{B}_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -s_{23} \\ s_{13} \\ s_{32} \\ -s_{31} \end{bmatrix}$$

Thus

$$u_1 = \begin{vmatrix} -s_{23} & a_{21} & a_{31} \\ s_{13} & a_{22} & a_{32} \\ s_{32} & b_{21} & b_{31} \end{vmatrix}; \quad u_2 = \begin{vmatrix} a_{11} & -s_{23} & a_{31} \\ a_{12} & s_{13} & a_{32} \\ b_{11} & s_{32} & b_{31} \end{vmatrix}; \quad u_3 = \begin{vmatrix} a_{11} & a_{21} & -s_{23} \\ a_{12} & a_{22} & s_{13} \\ b_{11} & b_{21} & s_{32} \end{vmatrix};$$

Furthermore, $Cv = TAv$ for all $v \in R^3$ also holds. Choose v such that v is perpendicular to the last row of A and C . Repeating above technique, one can derive:

$$v_1 = \begin{vmatrix} -t_{23} & a_{21} & a_{31} \\ t_{13} & a_{22} & a_{32} \\ t_{32} & b_{21} & b_{31} \end{vmatrix}; \quad v_2 = \begin{vmatrix} a_{11} & -t_{23} & a_{31} \\ a_{12} & t_{13} & a_{32} \\ b_{11} & t_{32} & b_{31} \end{vmatrix}; \quad v_3 = \begin{vmatrix} a_{11} & a_{21} & -t_{23} \\ a_{12} & a_{22} & t_{13} \\ b_{11} & b_{21} & t_{32} \end{vmatrix}$$

From

$$u_1 a_1 + u_2 a_2 + u_3 a_3 = 0$$

$$v_1 a_1 + v_2 a_2 + v_3 a_3 = 0$$

One can derive $(a_1 \ a_2 \ a_3)$ up to a scalar by taking the cross product of $(u_1 \ u_2 \ u_3)$ and $(v_1 \ v_2 \ v_3)$ where a_i are the depth component of A_i .

The last observation is to consider the following equation:

$$(1 \ 0 \ 0)^t = \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3$$

Then the following facts holds:

1. $\alpha_1, \alpha_2, \alpha_3$ are unique
2. $(\alpha_1 \ \alpha_2 \ \alpha_3)^t \cdot (a_{11} \ a_{21} \ a_{31})^t = 1$
3. $(\alpha_1 \ \alpha_2 \ \alpha_3)^t \cdot (a_{12} \ a_{22} \ a_{32})^t = 0$
4. $(\alpha_1 \ \alpha_2 \ \alpha_3)^t \cdot (a_1 \ a_2 \ a_3)^t = 0$

Since $(a_1 \ a_2 \ a_3)$ can be derived up to a scalar, one can compute $(\alpha_1 \ \alpha_2 \ \alpha_3)$ up to a scalar by taking cross product of $(a_{12} \ a_{22} \ a_{32})$ and $(a_{11} \ a_{21} \ a_{31})$. Next, one can compute the scalar of α_i 's by using fact (2). Now we know s_{11} and s_{12} by computing the first two components of $\alpha_1 B_1 + \alpha_2 B_2 + \alpha_3 B_3$.

If we write another unit vector $(0 \ 1 \ 0)$ in terms of A_i 's then one has

$$(0 \ 1 \ 0)^t = \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3$$

The following facts follows:

1. $\alpha_1, \alpha_2, \alpha_3$ are unique
2. $(\alpha_1 \ \alpha_2 \ \alpha_3)^t \cdot (a_{11} \ a_{21} \ a_{31})^t = 0$
3. $(\alpha_1 \ \alpha_2 \ \alpha_3)^t \cdot (a_{12} \ a_{22} \ a_{32})^t = 1$
4. $(\alpha_1 \ \alpha_2 \ \alpha_3)^t \cdot (a_1 \ a_2 \ a_3)^t = 0$

One could easily derive s_{21}, s_{22} , thus S can be computed.

Now we briefly write down the algorithm for the case of the same scale and then note the necessary change for the case of unknown scales.

Algorithm for three view of four points:

1. Compute *tem 1*, *tem 2*, *tem 3*, *tem 4* as in (A.1) (A.2), then the tilt direction is given by $(tem \ 2 + tem \ 4, tem \ 1 + tem \ 3)$.

2. $s_{31} = \eta \text{ tem } 4; s_{32} = \eta \text{ tem } 1; s_{13} = \delta \text{ tem } 3; s_{23} = \delta \text{ tem } 2$ where δ can be written in terms of η by requiring

$$\eta^2 (\text{tem } 4^2 + \text{tem } 1^2) = \delta^2 (\text{tem } 3^2 + \text{tem } 2^2)$$

The goal here is to adjust the magnitude of these two directional vectors to be the same. Add $(s_{31} \ s_{32})$ and $(s_{13} \ s_{23})$ and compare with the tilt direction which would give the correct pair of directional vectors. There is no need to find out η , one can assign any number at this stage.

3. Compute $u_1, u_2, u_3, v_1, v_2, v_3$ as (F.1)(F.2). This gives two vectors which are perpendicular to the vector formed by the depth components of A_i .
4. Compute the depth component by taking the vector product of (u_1, u_2, u_3) and (v_1, v_2, v_3) and denote it by (a_1, a_2, a_3) .
5. Taking the vector product of $(a_{12} \ a_{22} \ a_{32})$ and $(a_1 \ a_2 \ a_3)$ and denoted by $(\alpha_1 \ \alpha_2 \ \alpha_3)$. Scale $(\alpha_1 \ \alpha_2 \ \alpha_3)$ by requiring the inner product of $(\alpha_1 \ \alpha_2 \ \alpha_3)$ and $(a_{11} \ a_{21} \ a_{31})$ to be 1. Compute the first two components of $\alpha_1 B_1 + \alpha_2 B_2 + \alpha_3 B_3$ which gives s_{11} and s_{12} .
6. Take the vector product of $(a_{11} \ a_{21} \ a_{31})$ and $(a_1 \ a_2 \ a_3)$ and denote by $(\alpha_1 \ \alpha_2 \ \alpha_3)$. Scale $(\alpha_1 \ \alpha_2 \ \alpha_3)$ by requiring the inner product of $(\alpha_1 \ \alpha_2 \ \alpha_3)$ and $(a_{12} \ a_{22} \ a_{32})$ to be 1. Compute the first two components of $\alpha_1 B_1 + \alpha_2 B_2 + \alpha_3 B_3$ which gives s_{12} and s_{22} .

If one is only interested in knowing the the motion parameters of S , one can calculate the scale of the directional vectors easily since the norm of each row or column of a rotational matrix must be 1. From the rotational matrix, one can easily derive other motion parameters. These two solutions (due to the sign of directional vectors) explain the effect of the reflection of an object. If one uses the order of the second frame, third frame, and first frame and run the above algorithm, one would obtain $r_{11}, r_{12}, r_{21}, r_{22}$. If one uses the order of first frame, third frame and the second frame and run the above

algorithm, one would obtain $t_{11}, t_{12}, t_{21}, t_{22}$.

When the scale is unknown, the first step in the above algorithm does not give the tilt direction. One would perform the first and second step. From step 2, one would conclude that there are two possible tilts. Comparing these two possible tilts with $(tem4 + \sigma tem3, tem1 + \sigma tem2)$, one can then recover the scale. One would then incorporate the scale into step 5 and 6 to derive the correct $s_{11}, s_{12}, s_{21}, s_{22}$.

5. Example

Example 1: The first example consists of three views of four noncoplanar points with the same scale. The starting configuration of object points are as below. One then rotates this object by 30 degrees about an axis with 30 degrees of tilt and 30 degrees of slant. Second, one rotates again the object by 20 degrees about an axis with 45 degrees of tilt and 10 degrees of slant (See Figure 1) The observables are

$$\begin{aligned} A_1 &= (10.0 \ 20.0); A_2 = (20.0 \ 10.0); A_3 = (6.0 \ 54.0); \\ B_1 &= (2.8 \ 19.5); B_2 = (17.1 \ 13.9); B_3 = (-11.9 \ 44.2); \\ C_1 &= (-3.0 \ 18.7); C_2 = (12.4 \ 18.2); C_3 = (-24.0 \ 36.1); \end{aligned}$$

The rotational matrix for S, R, T are as follows:

$$S = \begin{bmatrix} 0.894 & -0.418 & 0.175 \\ 0.447 & 0.874 & -0.187 \\ -0.074 & 0.245 & 0.966 \end{bmatrix}; R = \begin{bmatrix} 0.940 & -0.335 & 0.049 \\ 0.337 & 0.940 & -0.034 \\ -0.034 & 0.049 & 0.998 \end{bmatrix}; T = \begin{bmatrix} 0.684 & -0.675 & 0.275 \\ 0.724 & 0.672 & -0.150 \\ -0.083 & 0.302 & 0.949 \end{bmatrix}$$

1. The first step yields 30 degrees of tilt for S and 45 degrees of tilt for R
2. Second step yields:

$$\begin{bmatrix} s_{31} \\ s_{32} \end{bmatrix} = \begin{bmatrix} -7.009 \\ 7.499 \end{bmatrix} = -40.5 \begin{bmatrix} 0.175 \\ -0.185 \end{bmatrix}; \begin{bmatrix} s_{13} \\ s_{23} \end{bmatrix} = \begin{bmatrix} 2.990 \\ -9.820 \end{bmatrix} = -40.4 \begin{bmatrix} -0.074 \\ 0.245 \end{bmatrix}$$

3. Through step 5, one obtains

$$\begin{bmatrix} s_{11} & s_{21} \\ s_{12} & s_{22} \end{bmatrix} = \begin{bmatrix} 0.891 & -0.418 \\ 0.447 & 0.874 \end{bmatrix}; \begin{bmatrix} r_{11} & r_{21} \\ r_{12} & r_{22} \end{bmatrix} = \begin{bmatrix} 0.940 & -0.335 \\ 0.337 & 0.940 \end{bmatrix}; \begin{bmatrix} t_{11} & t_{21} \\ t_{12} & t_{22} \end{bmatrix} = \begin{bmatrix} 0.684 & -0.675 \\ 0.724 & 0.672 \end{bmatrix}$$

4. It is obvious to see that the following two sets of solutions both account for the observables. One accounts for the movement of the reflection of the object.

$$\begin{aligned} S &= \begin{bmatrix} 0.894 & -0.418 & 0.175 \\ 0.447 & 0.874 & -0.187 \\ -0.074 & 0.245 & 0.966 \end{bmatrix}; R = \begin{bmatrix} 0.940 & -0.335 & 0.049 \\ 0.337 & 0.940 & -0.034 \\ -0.034 & 0.049 & 0.998 \end{bmatrix}; T = \begin{bmatrix} 0.684 & -0.675 & 0.275 \\ 0.724 & 0.672 & -0.150 \\ -0.083 & 0.302 & 0.949 \end{bmatrix} \\ S &= \begin{bmatrix} 0.894 & -0.418 & -0.175 \\ 0.447 & 0.874 & 0.187 \\ 0.074 & -0.245 & -0.966 \end{bmatrix}; R = \begin{bmatrix} 0.940 & -0.335 & -0.049 \\ 0.337 & 0.940 & 0.034 \\ 0.034 & -0.049 & -0.998 \end{bmatrix}; T = \begin{bmatrix} 0.684 & -0.675 & -0.275 \\ 0.724 & 0.672 & 0.150 \\ 0.083 & -0.302 & -0.949 \end{bmatrix} \end{aligned}$$

Example 2: The second example consists of the same three views of four noncoplanar

points as the first example with different scales. The relative scale of the second frame with respect to the first frame is 0.8 and that of the third frame is 0.6 (See Figure 3). The observables are

$$\begin{aligned} A_1 &= (10.0 \ 20.0); A_2 = (20.0 \ 10.0); A_3 = (6.0 \ 54.0); \\ B_1 &= (2.25 \ 15.6); B_2 = (13.71 \ 11.15); B_3 = (-9.59 \ 35.42); \\ C_1 &= (-1.84 \ 11.24); C_2 = (7.46 \ 10.92); C_3 = (-14.45 \ 21.68); \end{aligned}$$

Using step 1 and step 2, one recovers the scale 0.8 and 0.6 exactly. Therefore results should be the same as those in example 1. Other possible tilt would yield a negative scale which is thus excluded.

6. Conclusions

A new computational method to an augmented "four points three views" problem is introduced. The augmented problem might have unknown scales among these frames. We show that the tilt direction can be derived linearly while slant and rotational angle follow a two-staged linear computations. Detailed examples are provided to illustrate the method and a brief summarized algorithm is outlined also. Since the computations involve basically computing determinant of matrix, the theoretical error analysis (sensitivity) can be anticipated which is currently under investigation. In the second example of frames with different scales, it might happen that one additional scale would be introduced due to an additional tilt. However, we found that a negative scale is derived which leads the same unique recovery as example 1. It is also interesting to study whether a unique recovery for this augmented problem is possible while the additional tilt is simply an artifact.

Appendix

(I) The case that $(a \ b) = (0 \ 0)$ in Theorem 1.

This case occurs when $tem\ 1 = -tem\ 2$ and $tem\ 3 = -tem\ 4$. From Lemma 1, it is obvious that $|\eta| = |\delta|$. Thus one has

$$\begin{aligned} (s_{31} \ s_{32}) &= \eta (tem\ 3 \ tem\ 1) \\ (s_{13} \ s_{23}) &= \pm \eta (tem\ 4 \ tem\ 2) \\ &= \pm (tem\ 3 \ tem\ 1) \end{aligned}$$

One knows that $(s_{31} \ s_{32}) + (s_{13} \ s_{23})$ should be the tilt direction, thus one has $(tem\ 3, tem\ 1)$ as the tilt direction.

(II) The following case would not occur:

$$s_{31} = \eta \begin{vmatrix} \bar{A}_1 & \bar{A}_2 & \bar{A}_3 \\ b_{11} & b_{21} & b_{31} \end{vmatrix} = 0 \text{ and } s_{32} = \eta \begin{vmatrix} \bar{A}_1 & \bar{A}_2 & \bar{A}_3 \\ b_{12} & b_{22} & b_{32} \end{vmatrix} = 0$$

This situation implies that

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} \bar{A}_1 & \bar{A}_2 & \bar{A}_3 \end{bmatrix} = \begin{bmatrix} \bar{B}_1 & \bar{B}_2 & \bar{B}_3 \end{bmatrix}$$

This relation implies that the motion is degenerate [7] and is already excluded.

(III) The correct choice of the right hand side of the following system of equations is the second one.

$$\begin{bmatrix} \bar{A}_1 & \bar{A}_2 & \bar{A}_3 \\ \bar{B}_1 & \bar{B}_2 & \bar{B}_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -s_{23} \\ s_{13} \\ -s_{32} \\ s_{31} \end{bmatrix} \text{ or } \begin{bmatrix} -s_{23} \\ s_{13} \\ s_{32} \\ -s_{31} \end{bmatrix}$$

Consider the determinant of the matrix of the left hand side augmented by the first vector of the right hand side, one would find that it equals to $s_{23}^2 + s_{31}^2 \neq 0$.

References

1. Ullman, S. (1979), "The Interpretation of Visual Motion", MIT press.
2. Nagel H.H. and B. Neuman (1981), "On 3-D reconstruction from two perspective views", *Proc. IJCAI 81*, Vol II.
3. Roach, J.W. and J.K. Aggarwal, "Determining the movement of objects from a sequence of images", *IEEE Trans. PAMI 2*,
4. Tsai, R. Y. and T.S. Huang (1984), "Uniqueness and Estimation of Three-Dimensional Motion Parameters of Rigid Objects with Curved Surfaces", *IEEE Trans. PAMI 6*, 13-27.
5. Fang, J. and T.S. Huang (1984), "Some Experiments on Estimating the 3-D Motion Parameters of a Rigid Body from Two Consecutive Image Frames", *IEEE Trans. PAMI 6*, 545-554.
6. Rogers D.F. and J.A. Adams (1976), *Mathematical Elements for Computer Graphics*, McGraw-Hill, New York.
7. Lee, C.H. (1986), "One Correspondence, Motion, Scale, and Structure of Two Views of a Scene" TR-591, Department of Computer science, Purdue University.

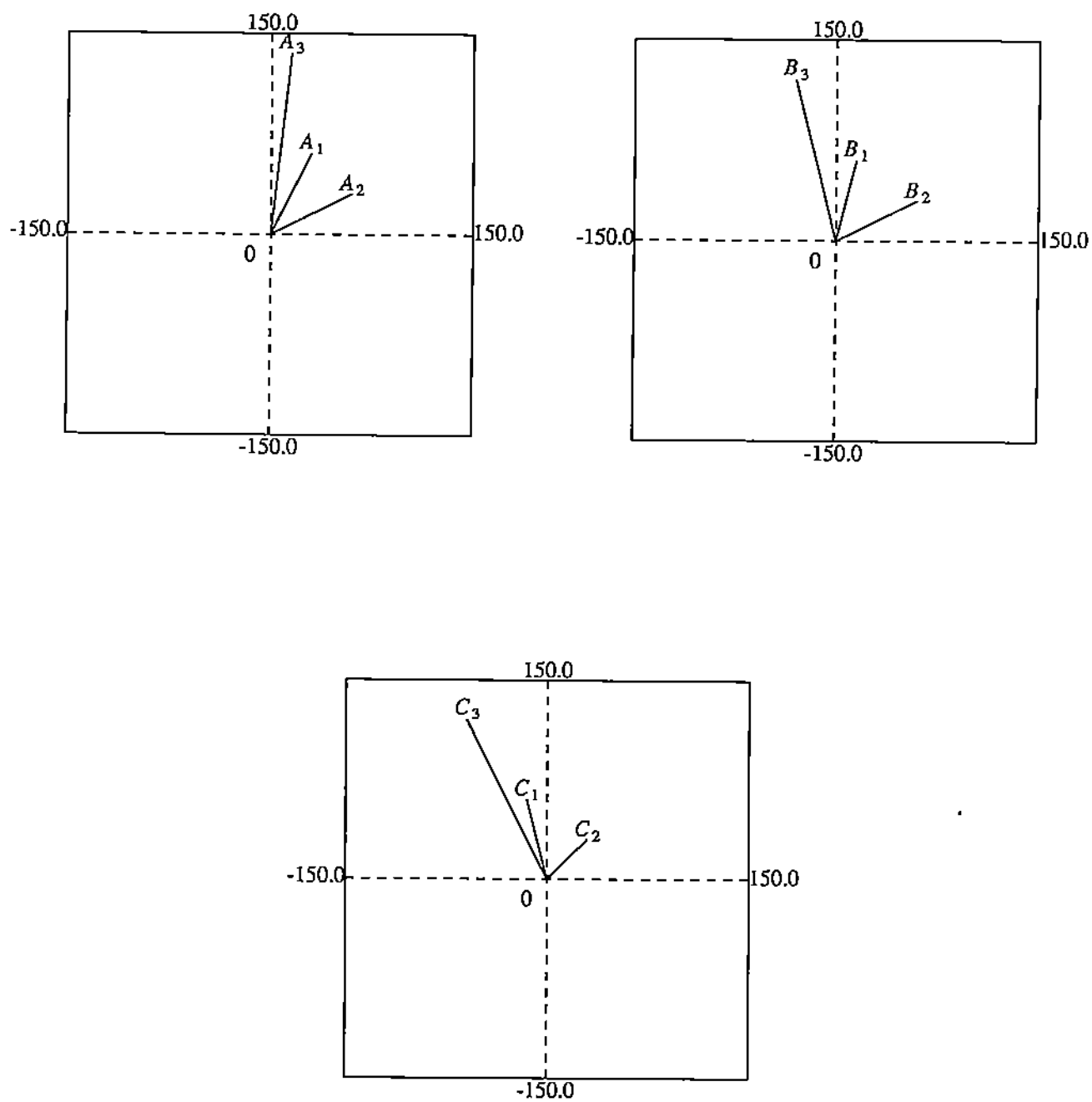


Figure 1: Three frames with the same scale.

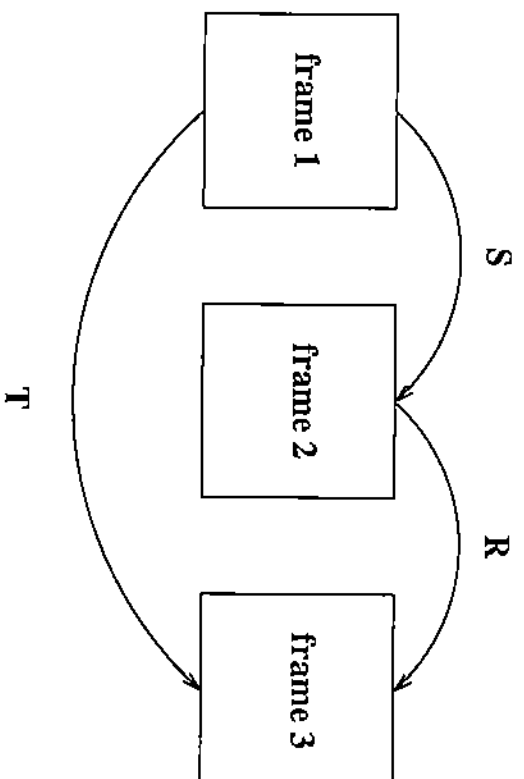


Figure 2

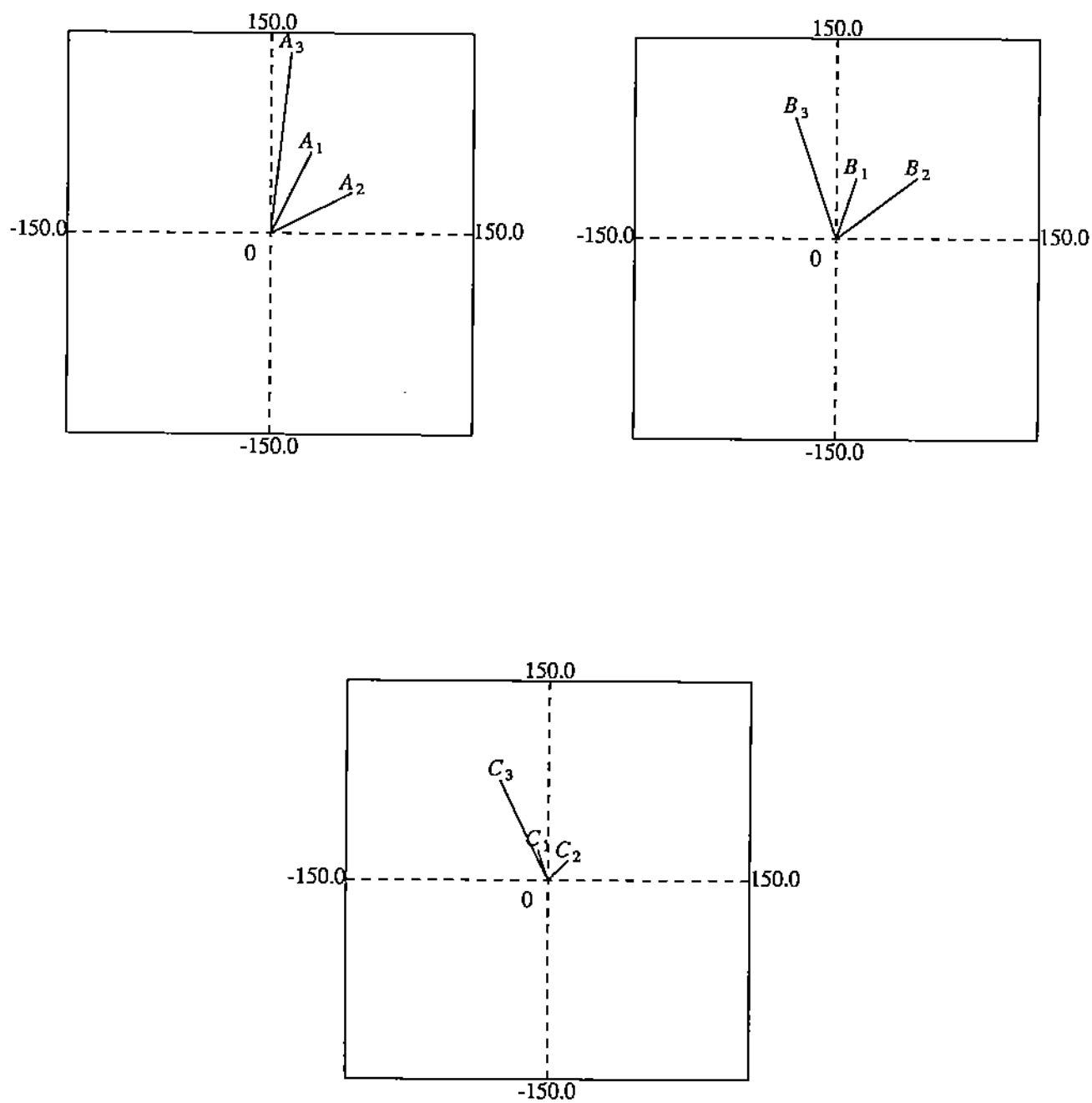


Figure 3: The scale for frame 1 is 1.0, the scale of frame 2 is 0.8, and the scale of frame 3 is 0.6.